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Growing Polyharmonic Functions and The Cauchy Problem for Some Domains in Two Dimensional Euclidean Space

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Abstract

In this paper we consider the Cauchy problem for the second-order polyharmonic equation in an unbounded domain D, where boundary data are given on a part of boundary. The goal is to reconstruct a function $v(\eta)$ in D based on the given values of v, its Laplacian, and their normal derivatives. This is the ill-posed inverse problem, hence we use Carleman-type integral representations and stability estimates to ensure well-posed approximations. We construct the Carleman function and prove key inequalities governing its behavior. Additionally, Lavrent'ev regularization is applied to construct an approximate solution. We establish a regularized solution of the Cauchy problem for the biharmonic equation in an unbounded domain. These results provide a foundation for stable reconstruction methods for polyharmonic functions in certain classes of unbounded domains.

Keywords: biharmonic functions; Carleman function; Phragmen-Lindelof type theorems; Cauchy problem; integral representation.

1 Introduction

Biharmonic functions naturally appear in elasticity theory, fluid mechanics, potential theory, and mathematical physics. Their study is essential for solving higher-order partial differential equations (PDEs) that describe physical phenomena such as elasticity, plate bending, and Stokes flows in fluid dynamics.

For arbitrary prescribed boundary data, if we consider the Cauchy problem with boundary conditions specified on a subset of the boundary, it is not possible to extend the values of the solution throughout the entire domain. This type of problem is ill-posed. Carleman [2] introduced formula for functions, which are bounded, proposing to modify the Cauchy classical integral formula by adding an auxiliary function which depends on some parameter. This approach allows, through a limiting process, to attenuate the impact of integral terms along the part of the boundary. Researches of Carleman remained largely unpursued for many years, but unstable problems continued to appear in applications. In 1943, Tikhonov [15, 14] identified the fundamental nature of ill-posed problems, emphasizing their practical significance and demonstrating that constraining the solution space to a compact set could stabilize these problems. These advancements led Lavrent'ev [9] to propose the idea constructing function of Carleman and apply it to regularize the solution of ill-posed problems. From 1955 in several works, Lavrent'ev [9] for some problems of mathematical physics pointed out how to select class of correctness.

In 1977, using ideas of Lavrent'ev, Yarmukhamedov [16] constructed explicitly a special fundamental solution for harmonic functions, when D_ρ is a simply connected bounded domain with boundary ∂D_ρ, consisting of the cone surface ∂G_ρ and the smooth part of the surface S lying in $\overline{G_\rho} = G_\rho \cup \partial G_\rho$, where $G_\rho = \left\{ y : y = (y', y_m), \ \ |y'| \le \tau y_m, \ \ \tau = \tan \dfrac{\pi}{2\rho}, \ \ \rho > 1 \right\}$. Later in [17], he obtained the following results.

Theorem 1.1. *Let for the function $u(y)$, where $D (D \subset \mathbb{R}^k)$ a simply connected domain of the half-space $y_k > 0$ and the following conditions hold:*

$$\begin{gathered}
\Delta u(y) = 0, \\
|\operatorname{grad} u(y)| + |u(y)| \le e^{|y|^\rho}, \qquad \text{where} \quad y \in D, \qquad \rho < 1, \\
\int_{\partial D} \frac{|u(y)|}{1+|y|^k} ds < \infty, \qquad \int_{\partial D} \left| \frac{\partial u}{\partial \vec{n}} \right| \frac{1}{1+|y|^{k-1}} ds < \infty.
\end{gathered}$$

Then, for all $\forall \ x_0 \in D$, the equality holds,

$$u(x_0) = C \int_{\partial D} \frac{u(y)}{r^k} ds.$$

In 1990, Ashurova [1] proved the following theorem,

Theorem 1.2. *If for harmonic function $u(y)$, defined in domain $D = \left\{ (y_1, y_2) : y_2 > y_1^2 \right\}$, the following conditions,*

$$\begin{gathered}
|\operatorname{grad} u(y)| + |u(y)| \le e^{|y|^\rho}, \qquad y \in D, \qquad \rho < 1, \\
u(y) = 0, \qquad \lim_{y \to \infty} \frac{\partial}{\partial \vec{n}} u(y) = 0, \qquad \forall \ y \in \partial D,
\end{gathered}$$

are fulfilled, then, $u(x) \equiv 0$.

In 2008, Juraeva [4] considered n ordered polyharmonic equations in certain unbounded domains (for all odd m, where m is a dimension of the space), meanwhile, Khasanov and Tursunov [7, 8] considered Cauchy problem for Laplace equation. It is worth to note that, Juraeva [5, 6] obtained the results for biharmonic functions in some domains of two and three dimensional Euclidean space. Besides, Sweis et al. [12] present a data-driven method for reconstructing complex functions from limited data, and Sweis et al. [13] analyzes the stability of solutions to higher-order differential equations. This work is devoted to the study of Cauchy problem for biharmonic equation.

2 Fundamental Concepts

Definition 2.1. *The function $v(\eta) = u(\eta_1, \eta_2, \ldots, \eta_k)$ defined in domain D of the Euclidean space $\mathbb{R}^k$, which has continuous partial derivatives up to $2p$ order (included), is called the p order polyharmonic function in D and the polyharmonic equation is defined as,*

$$\Delta^p v(\eta) = 0 \ \left(\Delta^p v(\eta) = \Delta(\Delta^{p-1} v(\eta))\right),$$

where Δ is the Laplace operator. For $p = 1$, this function is called harmonic function and for $p = 2$, second ordered polyharmonic function or biharmonic function.

Biharmonic functions are investigated from various prospectives in [10] and also some applications of biharmonic functions are studied in [11].

Definition 2.2. *The Carleman function for the point $\zeta \in D \subset \mathbb{R}^k$ and the part $\partial D \setminus S$ is the function $H_\sigma(\eta, \zeta)$ given for $\eta \neq \zeta$, $\sigma > 0$, provided it meets the following criteria:*

1. *$H_\sigma(\eta, \zeta)$ has the following form,*

$$H_\sigma(\eta,\zeta) = \begin{cases} C_{m,n}|\eta-\zeta|^{2n-k}\ln|\eta-\zeta| + G_\sigma(\eta,\zeta), & k \text{ is an even number, } 2n \geq k, \\ C_{m,n}|\eta-\zeta|^{2n-k} + G_\sigma(\eta,\zeta), & \text{in other cases,} \end{cases}$$

where

$$C_{k,n} = (-1)^{\frac{k}{2}-1}\left(2^{2n-1}\Gamma\left(\frac{2n-k}{2}\right)\pi^{\frac{k}{2}}\Gamma(n)\right)^{-1},$$

and $G_\sigma(\eta, \zeta)$ is the regular solution of polyharmonic equation of order n with respect to variable η and continuously differentiable on $D \cup \partial D = \overline{D}$.

2. *The inequality is satisfied for the fixed $\zeta \in D$ function $H_\sigma(\eta, \zeta)$,*

$$\sum_{m=0}^{n-1}\int_{\partial D\setminus S}\left(|\Delta^m H_\sigma(\eta,\zeta)| + \left|\frac{\partial \Delta^m H_\sigma(\eta,\zeta)}{\partial \vec{n}}\right|\right) ds_\eta \leq C(\zeta)\varepsilon(\sigma),$$

where $\vec{n}$ is the directional outer normal to ∂D, $\lim\limits_{\sigma\to\infty}\varepsilon(\sigma) = 0$ and polynomial $C(\zeta)$ depends only on ζ.

3 Statement of The Problem

Let $\mathbb{R}^2$ is a two dimensional real Euclidean space, $\zeta, \eta \in \mathbb{R}^2$, $\zeta' = (\zeta_1, 0)$, $\eta' = (\eta_1, 0)$, $r = |\zeta - \eta|$, $\alpha = |\zeta' - \eta'|$, $\alpha^2 = s$ and D is an unbounded simply connected domain lying inside layer,

$$\left\{ \eta : \eta = (\eta_1, \eta_2), \qquad 0 < \eta_2 < \frac{\pi}{\rho}, \qquad \rho > 0 \right\}.$$

The boundary $\partial D = S \cup L$, where $L = \{\eta : \eta_2 = 0\}$ and S satisfies the Lyapunov condition within a certain circle $F(0, \tilde{R})$ (circle of radius $\tilde{R}$ with the center at (0,0)) and outside the circle, it can be represented as a continuous function $\eta_2 = f(\eta_1)$ with bounded first-order partial derivatives and also

$$\int_{\partial D} \exp\left\{-b_0 \cosh(\rho_1 |\eta'|)\right\} ds < \infty, \qquad 0 < \rho_1 < \rho,$$

holds.

We consider the following problem. Let,

$$\Delta^2 v = 0, \qquad v \in C^4(D), \qquad \eta \in D, \tag{1}$$

$$v(\eta) = F_0(\eta), \qquad \Delta v(\eta) = F_1(\eta), \qquad \eta \in S, \tag{2}$$

$$\frac{\partial v(\eta)}{\partial \vec{n}} = G_0(\eta), \qquad \frac{\partial \Delta v(\eta)}{\partial \vec{n}} = G_1(\eta), \qquad \eta \in S, \tag{3}$$

where $F_i(\eta), G_i(\eta)$, $i = 0, 1$ are continuous functions given on S, $\vec{n}$ external normal to ∂D. It is required restoring $v(\eta)$ in D.

4 Results and Discussions

Let us define the function $H_\sigma(\eta, \zeta)$ by the equality,

$$H_\sigma(\eta, \zeta) = \frac{\exp\left(a \cosh\left(i\rho_1 \left(\zeta_2 - \frac{h}{2}\right)\right)\right)}{\exp(\sigma\zeta_2 + \zeta_2^2)} \int_{\sqrt{s}}^{\infty} \operatorname{Im}\left[\frac{\exp\left(\omega^2 + \sigma\omega\right)}{\exp\left(a \cosh\left(\rho_1 i\left(\omega - \frac{h}{2}\right)\right)\right)(\omega - \zeta_2)}\right](u^2 - s)du,$$

where $\omega = iu + \eta_2$, $s > 0$, $\sigma > 0$, $a \geq 0$, $0 < \rho_1 < \rho$ (for convenience, we designate all constant numbers by c_0 in the sequence).

The following results are extracted from Juraeva [6].

Theorem 4.1. [6] *The function $H_\sigma(\eta, \zeta)$ has the form,*

$$H_\sigma(\eta, \zeta) = c_0 \left(|\zeta - \eta|^2 \ln|\zeta - \eta| + |\zeta - \eta|^2 G_\sigma(\eta, \zeta)\right),$$

and $H_\sigma(\eta, \zeta)$, $|\zeta - \eta|^2 G_\sigma(\eta, \zeta)$ are second ordered polyharmonic functions respect to variable η ($\eta \neq \zeta$).

Let us introduce the notion,

$$A = -\sigma\eta_2 - \eta_2^2 + \sigma\zeta_2 + \zeta_2^2 + s + a\cosh\left(\rho_1\sqrt{s}\right)\cos\left(\rho_1\left(\eta_2 - \frac{h}{2}\right)\right).$$

Lemma 4.1. [6] *The function $H_\sigma(\eta,\zeta)$ satisfies the inequalities,*

$$
\begin{aligned}
|H_\sigma(\eta,\zeta)| &\le (\sigma|\zeta-\eta|+1)\frac{c_0}{\exp A}\\
\left|\frac{\partial}{\partial\eta_1}H_\sigma(\eta,\zeta)\right| &\le \left(1+\sigma+\sigma^2\right)\left(1+\frac{1}{|\zeta-\eta|}\right)\frac{c_0}{\exp A}\\
\left|\frac{\partial}{\partial\eta_2}H_\sigma(\eta,\zeta)\right| &\le \left(1+\sigma+\sigma|\zeta-\eta|+\sigma|\zeta-\eta|^2+\sigma^2|\zeta-\eta|+|\zeta-\eta|+\frac{\sigma^2}{|\zeta-\eta|}+\frac{1}{|\zeta-\eta|}\right)\frac{c_0}{\exp A}\\
\left|\frac{\partial}{\partial\vec{n}}H_\sigma(\eta,\zeta)\right| &\le \left(\left(1+\sigma+\sigma^2\right)\left(1+|\zeta-\eta|+\frac{1}{|\zeta-\eta|}\right)+\sigma|\zeta-\eta|^2\right)\frac{c_0}{\exp A}.
\end{aligned}
$$

Lemma 4.2. [6] *The $\Delta H_\sigma(\eta,\zeta)$ and $\dfrac{\partial\Delta H_\sigma(\eta,\zeta)}{\partial\vec{n}}$ satisfy the inequalities,*

$$
\begin{aligned}
|\Delta H_\sigma(\eta,\zeta)| &\le \left(\left(1+\sigma+\sigma^2\right)\left(1+\frac{1}{|\zeta-\eta|}+\frac{1}{|\zeta-\eta|^2}\right)+\sigma|\zeta-\eta|\right)\frac{c_0}{\exp A}\\
\left|\frac{\partial\Delta H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right| &\le \left(\left(1+\sigma+\sigma^2\right)\left(\frac{1}{|\zeta-\eta|}+\frac{1}{|\zeta-\eta|^2}+\frac{1}{|\zeta-\eta|^3}\right)+\sigma+\sigma|\zeta-\eta|+\sigma^2+\frac{\sigma^3}{|\zeta-\eta|^2}\right)\\
&\quad\times\frac{c_0}{\exp A}.
\end{aligned}
$$

Theorem 4.2. [6] *Function $H_\sigma(\eta,\zeta)$ is Carleman function for point $\zeta\in D$ and ∂D, that is*

$$
\int_{\partial D}\left(|H_\sigma(\eta,\zeta)|+\left|\frac{\partial H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right|+|\Delta H_\sigma(\eta,\zeta)|+\left|\frac{\partial\Delta H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right|\right)|ds|\le\frac{C(\zeta)}{\exp(\sigma A)},
$$

where $\vec{n}$ is outer normal to boundary ∂D, $C(\zeta)$ is the polynomial that depends only on ζ.

Let us designate the biharmonic functions defined in the domain D,

$$
v(\eta)=v(\eta_1,\eta_2)\in C^4(D)\cap C^3\left(\bar{D}\right),
$$

by $B_{\rho_2}(D)$, which maintain the condition,

$$
\sum_{m=0}^{1}\left(|\Delta^m v(\eta)|+|\operatorname{grad}\Delta^m v(\eta)|\right)\le c_0\exp(\mathrm{o}(\exp(\rho_2|\eta'|))),\qquad \rho_2<\rho_1,\qquad \eta\in D,
$$

where o represents Landau "little-o" symbol.

According to above results, we prove the following theorem,

Theorem 4.3. *Let the function $v\in B_{\rho_2}(D)$ at any point $\eta\in\partial D$ and*

$$
\sum_{m=0}^{1}\left(|\Delta^m v(\eta)|+\left|\frac{\partial\Delta^m v(\eta)}{\partial\vec{n}}\right|\right)\le c_0\exp\left(b\cos\frac{\rho_3(2\eta_2-h)}{2}\exp\rho_3|\eta'|\right),
$$

where $\rho_3<\rho_2<\rho_1<\rho$, then for all ζ_0 points of the domain D, we have

$$
v(\zeta_0)=\sum_{m=0}^{1}\int_{\partial D}\left(\Delta^m H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}-\Delta^{1-m}v(\eta)\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right)ds.
$$

Proof. As S is the Lyapunov surface inside the circle of radius $\tilde{R}$, there exist tangent plane and consequently the normal for arbitrary point lying inside a sphere of radius $\tilde{R}$, there is such a number $\tilde{\rho} > 0$, the same for all points S, that if we take the part of the surface $S \cap F(\eta_0, \tilde{R})$ that falls within the Lyapunov sphere with the centered at any point $\eta_0 \in S$ of radius $\tilde{\rho}$, then the lines parallel to the normal to S at the point $\eta_0 \in S$ intersect the surface no more than once, i.e. the part inside the Lyapunov sphere is given using $\eta_2 = f_1(\eta_1)$. Moreover, there exist numbers A and λ, $0 < \lambda \leq 1$ the same for entire surface S that for any two points $\eta_1, \eta_2 \in S$ the equality $\theta < A|\eta_1 - \eta_2|^{\lambda}$, where θ is angle between the normals to S at the points η_1 and η_2, and also if ds is an element of the surface area, then $ds < cd\eta_2$. Using the compactness properties of $S \cap F(\eta, \tilde{R})$, from an arbitrary covering of circles with radius $\tilde{\rho} > 0$, there exist a finite sub covering that covers $S \cap F(\eta, \tilde{R})$. Therefore, we prove the statement in the case when S is a surface given by $\eta_2 = f_1(\eta_1)$, where f_1 is a bounded, continuous function with bounded first-order partial derivatives.

We denote by $F(0, R)$, the disc centered at the point $(0, 0)$ with radius R, also $D \cap F(0, R) = D_R^+$, $D \cap CF(0, R) = D_R^-$, $CF(0, R)$ is complement of this circle. Let ζ_0 be an arbitrary point of domain D and the distance μ between the point ζ_0 and the boundary ∂D such that $\varepsilon < \mu$. We take a positive number R_1 such that $R_1 \geq |\zeta_0| + 1$. From the properties of the function $u \in B_{\rho_2}(D)$ and inequalities in Lemmas 4.1 and 4.2, it follows that the integral,

$$\sum_{m=0}^{1} \int_{\partial D} \left(\Delta^m H_\sigma(\eta, \zeta_0) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} - \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right) ds,$$

exists. Therefore, for every positive ε, there exists a positive number R_2 such that,

$$\left| \sum_{m=0}^{1} \int_{\partial D_R^-} \left(\Delta^m H_\sigma(\eta, \zeta_0) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} - \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right) ds \right|$$
$$\leq \sum_{m=0}^{1} \int_{\partial D_R^-} \left| \Delta^m H_\sigma(\eta, \zeta_0) + \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right| \left| \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} + \Delta^{1-m} v(\eta) \right| |ds| < \varepsilon.$$

Indeed, if we denote

$$\begin{aligned} L_1 &= \{\eta : \eta_2 = 0\} \cap CF(0, R), \\ L_2 &= \{\eta : \eta_2 = f(\eta_1)\} \cap CF(0, R), \\ L_3 &= \{\eta : |\eta| = R\}, \end{aligned}$$

moreover, using the inequality,

$$mn + pq \leq (m + p)(n + q),$$

where $m, n, p, q \in \mathbb{R}^+$, and for $b < a_1$, $\dfrac{\exp\left(b \cos\left(\rho_2(\eta_2 - \frac{h}{2}\right) \exp(\rho_2|\eta'|)\right)}{\exp\left(a_1 \cos\left(\rho_1(\eta_2 - \frac{h}{2}\right) \cosh(\rho_1 t)\right)} < 1$ and

$\rho_3 < \rho_2 < \rho_1 < \rho$, we have

$$\sum_{m=0}^{1} \int_{L_1} \left| \Delta^m H_\sigma(\eta, \zeta_0) + \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right| \left| \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} + \Delta^{1-m} v(\eta) \right| |ds|$$
$$< \int_{L_1} \exp\left(b \cos \rho_2 \left(\eta_2 - \frac{h}{2}\right) \exp(\rho_2|\eta'|)\right) \left(|H_\sigma(\eta, \zeta)| + \left|\frac{\partial H_\sigma(\eta, \zeta)}{\partial \vec{n}}\right|\right.$$
$$\left. + |\Delta H_\sigma(\eta, \zeta)| + \left|\frac{\partial \Delta H_\sigma(\eta, \zeta)}{\partial \vec{n}}\right|\right) |ds| \leq c_0 \int_0^\infty l(t) t dt.$$

Due to Lemmas 4.1 and 4.2, we have

$$\sum_{m=0}^{1}\left(|\Delta^m H_\sigma(\eta,\zeta)| + \left|\frac{\partial \Delta^m H_\sigma(\eta,\zeta)}{\partial \vec{n}}\right|\right)$$
$$\leq \left(1+\sigma+\sigma^2+\sigma r+\sigma r^2+\frac{\sigma}{r}+\frac{\sigma}{r^2}+\frac{\sigma}{r^3}+\frac{\sigma^2}{r}+\frac{\sigma^2}{r^2}+\frac{\sigma^2}{r^3}+\frac{\sigma^3}{r^2}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}\right)\frac{c_0}{\exp A},$$

where $r=|\eta-\zeta|$.

On L_1, we have $\eta_2=0$, $\rho_1\left(\frac{-h}{2}\right)=-\rho_1\left(\frac{\pi}{2\rho}\right)=\frac{-\pi}{2}\frac{\rho_1}{\rho}$, $\frac{\rho_1}{\rho}<1$, $\rho_1<\rho\cos\left(\rho_1\left(\frac{-h}{2}\right)\right)\neq 0$ and furthermore, based on $0\leq\zeta_2\leq\frac{\pi}{\rho}$, $\forall\, t>0$, $\zeta_2\leq\sqrt{t^2+\zeta_2^2}\leq t+\zeta_2$, we obtain

$$l(t)=\left(1+\sigma+\sigma^2+\sigma\sqrt{t^2+\zeta_2^2}+\sigma(t^2+\zeta_2^2)+\frac{\sigma^2+\sigma+1}{\sqrt{(t^2+\zeta_2^2)}}+\frac{\sigma^3+\sigma^2+\sigma+1}{(t^2+\zeta_2^2)}\right.$$
$$\left.+\frac{\sigma^2+\sigma+1}{(t^2+\zeta_2^2)^{\frac{3}{2}}}\right)\frac{c_0}{\exp A}$$
$$\leq\left(1+\sigma+\sigma^2+\sigma(t+\zeta_2)+\sigma\left(t^2+\zeta_2^2\right)+\frac{\sigma^2+\sigma+1}{\zeta_2}+\frac{\sigma^3+\sigma^2+\sigma+1}{\zeta_2^2}+\frac{\sigma^2+\sigma+1}{\zeta_2^3}\right)\frac{c_0}{\exp A}.$$

Using the equalities,

$$\int_0^\infty\frac{\zeta^n d\zeta}{\exp(r^2\zeta)}=\frac{n!}{r^{2n+2}},\qquad \int_0^\infty\frac{dt}{\sqrt{r^2t+s}\exp(r^2t+s)}=\frac{\sqrt{\pi}}{2r^2},$$
$$\int_0^\infty\frac{dt}{\exp(r^2t+s)}=\frac{c_0}{r^2},\qquad \int_0^\infty\frac{t^{p-\frac{1}{2}}dt}{\exp(at)}=\frac{(2p-1)!!}{2^p}\frac{\sqrt{\pi}}{a^{p+\frac{1}{2}}},$$

for $a>0, p=1,2,\ldots$, therefore, for $\varepsilon>0$, there exists $P_1>0$, such that,

$$\sum_{m=0}^{1}\int_{L_1}\left|\Delta^m H_\sigma(\eta,\zeta_0)+\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right|\left|\frac{\partial\Delta^{(1-m)}v(\eta)}{\partial\vec{n}}+\Delta^{(1-m)}v(\eta)\right||ds|<\frac{\varepsilon}{3}.$$

The evaluation of the integral over L_2 can be obtained in a similar manner. Therefore, for the same $\varepsilon>0$, there exists $P_2>0$, such that,

$$\sum_{m=0}^{1}\int_{L_2}\left|\Delta^m H_\sigma(\eta,\zeta_0)+\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right|\left|\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}+\Delta^{1-m}v(\eta)\right||ds|\leq\frac{\varepsilon}{3}.$$

Also on L_3, we have

$$\sum_{m=0}^{1}\int_{L_3}\left|\Delta^m H_\sigma(\eta,\zeta_0)+\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right|\left|\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}+\Delta^{1-m}v(\eta)\right||ds|$$
$$\leq c_0\int_0^\infty\exp(\mathrm{o}\exp(\rho_1|\eta'|))l_1(t)tdt,$$

where

$$l_1(t)=\left(\left(t^2+\zeta_2^2\right)t^{-1}+\left(t^2+\zeta_2^2\right)^{\frac{1}{2}}t^{-1}+t^{-1}+c_0\right)\exp(\sigma\eta_2-\sigma\zeta_2-A_2),$$
$$A_2=a_1\cosh(\rho_1 t)\cos\left(\rho_1\left(\eta_2-\frac{h}{2}\right)\right).$$

Choosing a_1 such that,

$$\frac{\exp\left(o\left(\exp(\rho_1|\eta'|)\right)\right)}{\exp\left(a_1\cos\left(\rho_1\left(\eta_2-\frac{h}{2}\right)\right)\cosh(\rho_1 t)\right)}<1,$$

the inequality holds, for the same $\varepsilon>0$, there exists $P_3>0$, such that,

$$\sum_{m=0}^{1}\int_{L_3}\left|\Delta^m H_\sigma(\eta,\zeta_0)+\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right|\left|\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}+\Delta^{1-m}v(\eta)\right||ds|<\frac{\varepsilon}{3}.$$

If we denote $P_4=\max(P_1,P_2,P_3)$, then for $P>P_4$, we obtain that,

$$\sum_{m=0}^{1}\left|\int_{\partial(D\cap CF(0,P))}\left(\Delta^m H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}-\Delta^{1-m}v(\eta)\frac{\partial\Delta^m H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right)ds\right|\le\varepsilon.$$

Let $\min(\mu,P)=R$, applying the Gutsmer's formula [3],

$$\iint_D(v\Delta^n u-u\Delta^n v)d\tau=\sum_{m=0}^{n-1}\int_{\partial D}\left(\Delta^m v\frac{\partial\Delta^{n-m-1}u}{\partial\vec{n}}-\Delta^{n-m-1}u\frac{\partial\Delta^m v}{\partial\vec{n}}\right)ds,$$

to the functions $H_\sigma(\eta,\zeta)$ and $v(\eta)$ in domain $G=D_R^+\setminus F(\zeta_0,\varepsilon)$, we have

$$\begin{aligned}&\iint_G(v(\eta)\Delta^2 H_\sigma(\eta,\zeta_0)-H_\sigma(\eta,\zeta_0)\Delta^2 v(\eta))d\tau\\&\quad=\sum_{m=0}^{1}\int_{\partial G}\left(\Delta^m v(\eta)\frac{\partial\Delta^{1-m}H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}-\Delta^{1-m}H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^m v(\eta)}{\partial\vec{n}}\right)ds.\end{aligned}$$

According to the functions $H_\sigma(\eta,\zeta)$ and $v(\eta)$ are biharmonic and

$$H_\sigma(\eta,\zeta)=c_0\left(|\zeta-\eta|^2\ln|\zeta-\eta|+|\zeta-\eta|^2G_\sigma(\eta,\zeta)\right),$$

then, we have

$$\begin{aligned}&\sum_{q=0}^{1}\int_{\partial D_R^+}\left(\Delta^q H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{(1-q)}v(\eta)}{\partial\vec{n}}-\Delta^{(1-q)}v(\eta)\frac{\partial\Delta^q H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right)ds\\&\quad=\sum_{q=0}^{1}\int_{\partial F(\zeta_0,\varepsilon)}\left(\Delta^q H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{(1-q)}v(\eta)}{\partial\vec{n}}-\Delta^{(1-q)}v(\eta)\frac{\partial\Delta^q H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right)ds.\end{aligned}$$

Now, we introduce the following notions,

$$\begin{aligned}M(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))&=\sum_{q=0}^{1}\int_{\partial F(\varepsilon,\zeta_0)}\Delta^q H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{(1-q)}v(\eta)}{\partial\vec{n}}ds\\&\quad-\sum_{q=0}^{1}\int_{\partial F(\varepsilon,\zeta_0)}\Delta^{(1-q)}v(\eta)\frac{\partial\Delta^q H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}ds,\\M^1(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))&=\sum_{q=0}^{1}\int_{\partial F(\varepsilon,\zeta_0)}\Delta^q H_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{(1-q)}v(\eta)}{\partial\vec{n}}ds,\\M^2(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))&=\sum_{q=0}^{1}\int_{\partial F(\varepsilon,\zeta_0)}\Delta^{(1-q)}v(\eta)\frac{\partial\Delta^q H_\sigma(\eta,\zeta_0)}{\partial\vec{n}}ds.\end{aligned}$$

We prove that $M(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))\to v(\zeta_0)$, as $\varepsilon\to 0$. Indeed,

$$\sum_{q=0}^{1}\int_{\partial F(\varepsilon,\zeta_0)}\Delta^q\left(|\zeta-\eta|^2\ln|\zeta-\eta|+|\zeta-\eta|^2G_\sigma(\eta,\zeta)\right)\frac{\partial\Delta^{(1-q)}v(\eta)}{\partial|\zeta-\eta|}ds$$
$$=\int_{\partial F(\varepsilon,\zeta_0)}\left(|\zeta-\eta|^2\ln|\zeta-\eta|+|\zeta-\eta|^2G_\sigma(\eta,\zeta)\right)\frac{\partial\Delta v(\eta)}{\partial|\zeta-\eta|}ds$$
$$+\int_{\partial F(\varepsilon,\zeta_0)}\Delta\left(|\zeta-\eta|^2\ln|\zeta-\eta|+|\zeta-\eta|^2G_\sigma(\eta,\zeta)\right)\frac{\partial v(\eta)}{\partial|\zeta-\eta|}ds.$$

Taking into account that the function $|\zeta-\eta|^2G_\sigma(\eta,\zeta)$ is harmonic and equality,

$$\Delta|\zeta-\eta|^p\ln|\zeta-\eta|=(p+2k-2)\left(p|\zeta-\eta|^{p-2}\ln|\zeta-\eta|+|\zeta-\eta|^{p-2}\right),$$

and inequality $\ln|\zeta-\eta|\le|\zeta-\eta|, \forall\ |\zeta-\eta|>0$, we write,

$$M^1(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))|$$
$$\le\int_{\partial F(\varepsilon,\zeta_0)}\left[\left(|\zeta-\eta|^3+|\zeta-\eta|^2G_\sigma(\eta,\zeta)\right)\frac{\partial\Delta v(\eta)}{\partial|\zeta-\eta|}\right]ds+\int_{\partial F(\varepsilon,\zeta_0)}c_0|\zeta-\eta|\frac{\partial v(\eta)}{\partial|\zeta-\eta|}ds$$
$$\le\int_{\partial F(\varepsilon,\zeta_0)}\left[\left(\varepsilon^3+\varepsilon^2G_\sigma(\eta,\zeta)\right)\frac{\partial\Delta v(\eta)}{\partial|\zeta-\eta|}+c_0\varepsilon\frac{\partial v(\eta)}{\partial|\zeta-\eta|}ds\right].$$

By the mean value theorem, there exists ξ and τ such that,

$$M^1(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))\le\left(\varepsilon^4+\varepsilon^3G_\sigma(\eta,\zeta)\right)\left[\frac{\partial\Delta v(\eta)}{\partial|\zeta-\eta|}\right](\xi)+c_0\varepsilon^2\left[\frac{\partial v(\eta)}{\partial|\zeta-\eta|}\right](\tau),$$

and therefore $M^1(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))\to 0$. Similarly for $M^2(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))$, we have

$$M^2(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))=\varepsilon^2[\Delta v](\sigma)+v(\zeta_0)+\varepsilon^{3-2q}\left[G_\sigma\Delta^{(1-q)}v\right](\nu),$$

and therefore, $M^2(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))\to v(\zeta_0)$. This implies $M(F(\zeta_0,\varepsilon),v,H_\sigma(\eta,\zeta_0))\to v(\zeta_0)$, that is, obtained an integral representation for biharmonic functions,

$$v(\zeta_0)=\sum_{m=0}^{1}\int_{\partial D}\left(\Delta^mH_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}-\Delta^{1-m}v(\eta)\frac{\partial\Delta^mH_\sigma(\eta,\zeta_0)}{\partial\vec{n}}\right)ds.$$

□

With the help of this integral representation obtained theorems of the Phragmen-Lindeleof type, which are subject of works of Juraeva [5, 6].

In a similar fashion, we obtain the following theorem,

Theorem 4.4. *Let these conditions be satisfied for a function $v\in B_{\rho_2}(D)$,*

$$\Delta^mv(\eta)=0,\quad m=0,1;\quad\sum_{m=0}^{1}\int_{\partial D}\left|\frac{\partial\Delta^mv(\eta)}{\partial\vec{n}}\right|ds<c_0,\quad\eta\in\partial D,$$

then, for all $\zeta_0\in D$, we have

$$v(\zeta_0)=\sum_{m=0}^{1}\int_{\partial D}\Delta^mH_\sigma(\eta,\zeta_0)\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}ds.$$

Let,

$$v_\sigma(\zeta) = \sum_{m=0}^{1} \int_S \left(G_{1-m} \Delta^m H_\sigma(\eta, \zeta) - F_{1-m} \frac{\partial \Delta^m H_\sigma(\eta, \zeta)}{\partial \vec{n}} \right) ds,$$

then, the following statement holds.

Corollary 4.1. *Let $v(\zeta)$ be the solution of problems (1)–(3) and the following conditions be satisfied,*

1. $v \in B_{\rho_2}(D)$.
2. *For arbitrary $\eta \in S$,*

$$\sum_{m=0}^{1} \left(|\Delta^m v(\eta)| + \left| \frac{\partial \Delta^m v(\eta)}{\partial \vec{n}} \right| \right) \le c_0 \exp \left(b \cos \left(\rho_3 \left(\eta_2 - \frac{h}{2} \right) \right) \exp \rho_3 |\eta'| \right).$$

3. $\sum_{m=0}^{1} \left(|\Delta^m v(\eta)| + \left| \frac{\partial \Delta^m v(\eta)}{\partial \vec{n}} \right| \right) \le M, \quad \forall\, \eta \in L$, *where M is a constant, then we have*

$$|v(\zeta)| \le M \varepsilon(\sigma) \exp(-A), \qquad \sigma \ge \sigma_0 > 0, \qquad \zeta \in D,$$

where $\varepsilon(\sigma)$ is the polynomial with defined coefficients.

Proof. From Theorem 4.3 and the definition of $v_\sigma(\zeta), \forall\, \zeta \in D$, we have

$$v(\zeta) = v_\sigma(\zeta) + \sum_{m=0}^{1} \int_{\eta_2=0} \left(\Delta^m H_\sigma(\eta, \zeta_0) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} - \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right) ds. \tag{4}$$

Since,

$$\begin{aligned} |v_\sigma(\zeta)| &\le \sum_{m=0}^{1} \left| \int_S \left(\Delta^m H_\sigma(\eta, \zeta_0) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} - \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right) ds \right| \\ &\le M \varepsilon(\sigma) \exp \left(\sigma \eta_2 - \sigma \zeta_2 - a \cosh(\rho_1 \alpha) \cos \left(\frac{\rho_1 h}{2} \right) \right), \end{aligned}$$

we have for $\forall\, \zeta \in D$,

$$\begin{aligned} |v(\zeta)| &\le |v_\sigma(\zeta)| + \sum_{m=0}^{1} \int_{\eta_2=0} \left| \Delta^m H_\sigma(\eta, \zeta_0) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} - \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta_0)}{\partial \vec{n}} \right| ds \\ &\le M \varepsilon(\sigma) \exp(-A). \end{aligned}$$

□

Taking the limit of (4) we have the following result.

Corollary 4.2. *The limit equality,*

$$\lim_{\sigma \to \infty} v_\sigma(\zeta) = v(\zeta),$$

holds uniformly in any compact set from D.

Theorem 4.5. *Let the condition of Corollary 4.1 be satisfied. Then, for any $\zeta \in D$ and $\sigma > 0$, the estimates hold,*

$$|v(\zeta) - v_\sigma(\zeta)| \leq c(\sigma, \zeta_2) M e^{-\sigma \zeta_2^2},$$

$$\left| \frac{\partial v(\zeta)}{\partial \zeta_i} - \frac{\partial v_\sigma(\zeta)}{\partial \zeta_i} \right| \leq c_i(\sigma, \zeta_2) M e^{-\sigma \zeta_2^2}, \qquad i = 1, 2,$$

where $c(\sigma, \zeta_2)$ and $c_i(\sigma, \zeta_2)$ are polynomials.

5 Regularization to Lavrent'ev

We considered the problem (1)–(3) and obtained the limiting equality $\lim\limits_{\sigma\to\infty} v_\sigma(\zeta) = v(\zeta)$ which holds uniformly in any compact set from D. Now, let us consider the problem when, along with $F_i(\eta), G_i(\eta)$, $i = 0, 1$, their approximations with a given deviation δ, $\delta \in (0, 1)$, $F_i^\delta(\eta)$, $G_i^\delta(\eta)$ are provided. It is required to restore $v(\eta)$ in D. Denoting

$$v_{\sigma\delta}(\zeta) = \sum_{m=0}^{1} \int_S \left(G_m^\delta \Delta^m H_\sigma(\eta, \zeta) - F_m^\delta \frac{\partial \Delta^m H_\sigma(\eta, \zeta)}{\partial \vec{n}} \right) ds,$$

we obtain following theorem.

Theorem 5.1. *Let $v(\zeta)$ a solution of the problem (1)–(3), and the following conditions be satisfied,*

1. *$v \in B_{\rho_2}(D)$.*
2. *For arbitrary $\eta \in S$,*

$$\sum_{m=0}^{1} \left(|\Delta^m v(\eta)| + \left| \frac{\partial \Delta^m v(\eta)}{\partial \vec{n}} \right| \right) \leq c_0 \exp \left(b \cos \left(\rho_3 \left(\frac{2\eta_2 - h}{2} \right) \right) \exp \rho_3 |\eta'| \right), b < a,$$

   where $\rho_3 < \rho_2 < \rho_1 < \rho$.

3. $\sum\limits_{m=0}^{1} \left(|\Delta^m v(\eta)| + \left| \dfrac{\partial \Delta^m v(\eta)}{\partial \vec{n}} \right| \right) \leq M, \forall\ \eta \in L$, *where M is a constant number, then for arbitrary $\zeta \in D$, we have*

$$|v(\zeta) - v_{\sigma\delta}(\zeta)| \leq P(\sigma) M^{1 - \frac{\zeta_2}{\zeta_2^0}} \delta^{\frac{\zeta_2}{\zeta_2^0}}, \qquad \sigma \geq \sigma_0 > 0, \qquad \zeta_2^0 = \max_{\zeta \in S} \zeta_2,$$

   where $\sigma = \dfrac{1}{\zeta_2^0} \ln \dfrac{M}{\delta}$, $M > \delta$, $P(\sigma)$ is a polynomial.

Proof. Let $F(v, \zeta, \sigma, \delta) = v(\zeta) - v_{\sigma\delta}(\zeta)$, then $\forall\ \zeta \in D$, we have

$$F(v, \zeta, \sigma, \delta) = \sum_{m=0}^{1} \int_{\partial D} \Delta^m H_\sigma(\eta, \zeta) \frac{\partial \Delta^{1-m} v(\eta)}{\partial \vec{n}} ds - \sum_{m=0}^{1} \int_{\partial D} \Delta^{1-m} v(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta)}{\partial \vec{n}} ds$$
$$- \sum_{m=0}^{1} \int_S \left[G_{1-m}^\delta(\eta) \Delta^m H_\sigma(\eta, \zeta) - F_{1-m}^\delta(\eta) \frac{\partial \Delta^m H_\sigma(\eta, \zeta)}{\partial \vec{n}} \right] ds,$$

$$F(v,\zeta,\sigma,\delta)=\sum_{m=0}^{1}\int_S\left(\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}-G^{\delta}_{1-m}(\eta)\right)\Delta^m H_\sigma(\eta,\zeta)ds$$
$$-\sum_{m=0}^{1}\int_S\left(\Delta^{1-m}v(\eta)-F^{\delta}_{1-m}(\eta)\right)\frac{\partial\Delta^m H_\sigma(\eta,\zeta)}{\partial\vec{n}}ds$$
$$-\sum_{m=0}^{1}\int_{\eta_2=0}\left[\Delta^m H_\sigma(\eta,\zeta)\frac{\partial\Delta^{1-m}v(\eta)}{\partial\vec{n}}-\Delta^{1-m}v(\eta)\frac{\partial\Delta^m H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right]ds.$$

As $\delta>0$ and $\sup\limits_S|F_i^\delta(\eta)-F_i(\eta)|<\delta$, $\sup\limits_S|G_i^\delta(\eta)-G_i(\eta)|<\delta$, we obtain

$$|F(v,\zeta,\sigma,\delta)|\le\delta\sum_{m=0}^{1}\left|\int_S\left(\Delta^m H_\sigma(\eta,\zeta)-\frac{\partial\Delta^m H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right)ds\right|$$
$$+M\sum_{m=0}^{1}\left|\int_S\left(\Delta^m H_\sigma(\eta,\zeta)-\frac{\partial\Delta^m H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right)ds\right|.$$

Due to estimates of functions $|\Delta^m H_\sigma(\eta,\zeta)|$, $\left|\frac{\partial\Delta^m H_\sigma(\eta,\zeta)}{\partial\vec{n}}\right|$, we have

$$|v(\zeta)-v_{\sigma\delta}(\zeta)|\le P(\sigma)M^{1-\frac{\zeta_2}{\zeta_2^0}}\delta^{\frac{\zeta_2}{\zeta_2^0}},\qquad\sigma\ge\sigma_0>0,\qquad\zeta_2^0=\max_{\zeta\in S}\zeta_2,$$

where $\sigma=\frac{1}{\zeta_2^0}\ln\frac{M}{\delta}$, $M>\delta$, $P(\sigma)$ is a polynomial. □

Corollary 5.1. *For any $\zeta\in D$, the limit equality,*

$$\lim_{\delta\to 0}v_{\sigma\delta}(\zeta)=v(\zeta),$$

holds uniformly in any compact set from D.

6 Conclusion

In this study, we investigated the Cauchy problem for second-order polyharmonic equations in unbounded domains, emphasizing stability and reconstruction methods. Using the constructed Carleman function and its estimates some properties of biharmonic funtions are obtained for an unbounded domains lying in the strip in $\mathbb{R}^2$. Moreover, an integral representation for biharmonic functions in the class $B_{\rho_2}(D)$. These results contribute to the broader mathematical framework for solving inverse problems involving polyharmonic functions, with potential applications in physics and engineering.

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